6.5 Stationary measures

μ is a measure on S. (countable)

$$\sum \mu(x) p(x,y) = \mu(y). \qquad \text{—} \; \star 1$$

$$\boxed{\mathbb{P}_\mu(X_1 = y) = \mu(y)}$$ ooh!

By induction,

$$\mathbb{P}_\mu(X_n = y) = \mu(y).$$

Question: How would you state this on a general state space S?

So if you start the chain with μ (st. $\mu(S)=1$, a probability distribution)

Then $P_\mu(X_n \in A) = \mu(A)$; i.e., the distribution of the n^{th} step is invariant.

We generally say μ is stationary for X if it satisfies $\star 1$ even if $\mu(S) = +\infty$.

Random walk: $S = \mathbb{Z}^d$. d-dimensional

$p(x,y) = f(y-x)$ where $f(z) \geq 0$.
$\sum f(z) = 1$

$$\sum_x \mu(x)\, p(x,y)$$
$$= \sum_x \mu(x)\, f(y-x)$$; simply set $\mu(x) = 1$.

Does this look like a Harmonic function?

A Martingale?

Recall that Harmonic functions for a countable chain are defined as follows: We say f is harmonic for the chain if

$$f(S_0) = f(x) = \sum p(x,y)\, f(y) = E_x[f(S_1)]$$

However μ is different:

$$\sum_x \mu(x)\, p(x,y) = \mu(y)$$

So μ is harmonic for the chain run in reverse!

with transition probability $\hat{p}(y,x) = p(x,y)$.

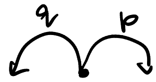
If $$\sum_x p(x,y) = \sum_x \hat{p}(y,x) = 1 \qquad \text{— } \star 2$$

Then the chain can be run in reverse since $\hat{p}(y,x)$ is a bonafide transition prob.

$p(x,y)$ satisfying $(\star 2)$ are called <u>doubly stochastic</u>

Ex: $p(x,y)$ is <u>doubly stochastic</u> $\Leftrightarrow$

$\mu(x) = 1$ is <u>a</u> stationary measure.

Over all of $\mathbb{Z}$ 

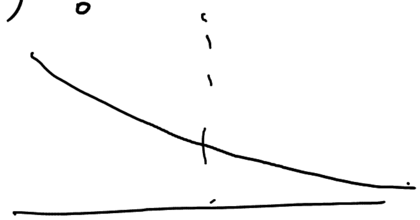
$p(x, x+1) = p \quad p(x, x-1) = q$

$\Rightarrow p$ is doubly stochastic

$\Rightarrow \mu = 1$ stationary measure

<u>Asymmetric random walk: simply</u> check $\mu(x) = (p/q)^x$.

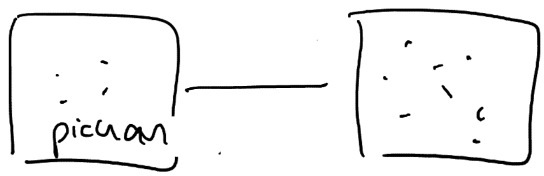
$S = \mathbb{Z}$



Easy to verify.

<u>Ehrenfest chain</u>

$S = \{0, 1, \cdots r\}$ (r balls in the 2 urns)



pick an urn at random & move it

<u>Example</u> <u>6.5.3</u> "Mixing".

Clearly $\mu(x)$ a stationary measure if you just randomly select a ball & put into one of the urns.

~~Let~~ Q: What is the Markov Chain here?

Fix an urn 1, and let X_n = # of balls in urn 1.

The stationary measure is very natural:

$S = \{1, \ldots, r\}$

$\downarrow$

$c_1, \ldots, c_r$ (coins)

Flip a coin to determine which urn it goes into.
Then $\mu(X_0 = k) = \binom{r}{k} \frac{1}{2^k}$ (k balls in Urn 1)

<u>Detailed Balance</u>: A measure μ satisfies detailed balance if

$$\mu(x)\, p(x,y) = \mu(y)\, p(y,x)$$

Such a measure μ is also called REVERSIBLE

Note that p does not need to be double stochastic (the CHAIN need not be reversible)

If μ is reversible then μ is stationary.

$$\sum_x \mu(x)\, p(x,y) = \sum_x \mu(y)\, p(y,x) = \mu(y)$$

Ex: Show that if S is irreducible and μ is stationary then $\mu(y) > 0 \ \forall y \in S$

if $\mu(y) = 0$, then $\sum \mu(x) p(x,y) = 0$.

This means that $\mu(x) = 0 \ \forall x$ st $p(x,y) > 0$

You can also iterate these things to get

Ex: $\sum \mu(x) p^n(x,y) = \mu(y)$

And once you do this you can simply use irreducibility.

Theorem 5.5.5 Let μ be a stationary measure and let P_μ be a Markov measure on $\{X_n\}_{n=0}^{\infty}$.

Then $Y_m = X_{n-m}$ $0 \le m \le n$ is a Markov chain with initial measure μ and transition

$$q(x,y) = \frac{\mu(y)\, p(y,x)}{\mu(x)}$$

Verify: $\sum_y q(x,y) = \sum \frac{\mu(y)\, p(y,x)}{\mu(x)} = \frac{\mu(x)}{\mu(x)} = 1$

$P(Y_{m+1} = y \mid Y_m = x) = q(x,y)$

$= P(X_{n-m-1} = y \mid X_{n-m} = x)$

$$= \frac{\overbrace{P(X_{n-m} = x \mid X_{n-m-1} = y)}^{p(y,x)} \overbrace{P(X_{n-m-1} = y)}^{\mu(y)}}{\underbrace{P(X_{n-m} = x)}_{\mu(x)}}$$

since μ is stationary.

If μ is a reversible measure, then

$$q(x,y) = \frac{\mu(y)\, p(y,x)}{\mu(x)} = p(x,y)$$

If p is doubly stochastic, then with $\mu = 1$
$q(x,y) = p(y,x)$

Reversible measures μ may not always exist.

This is called Kolmogorov's criterion, and is in Le Gall's exercises.

Theorem (When do reversible measures μ exist?)

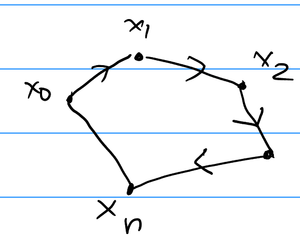
Suppose p is irreducible measures exist. A necessary and sufficient condition is

(i) $p(x,y) > 0 \Rightarrow p(y,x) > 0$

(ii) $\forall$ loop $x_0, x_1, \dots, x_n = x_0$ with

$$\prod_{1 \le i \le n} p(x_i, x_{i-1}) > 0$$ that you can traverse in one direction

$$\prod_{i=1}^{n} \frac{\overbrace{p(x_{i-1}, x_i)}^{\text{forward product}}}{p(x_i, x_{i-1})} = 1$$



Pf: If a reversible μ exists, then

$$\prod_{i=1}^{n} \frac{p(x_{i-1}, x_i)}{p(x_i, x_{i-1})} = \prod_{i=1}^{n} \frac{\cancel{p(x_i, x_{i-1})}\mu(x_i)}{\cancel{p(x_i, x_{i-1})}\,\mu(x_{i-1})}$$

$= 1$ (since we're on a loop.)

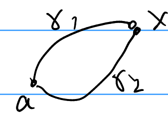
If the loop condition is satisfied, then

Pick $a \in S$, set $\mu(a) = 1$ and let

$$\mu(x) = \prod_{i=1}^{n} \frac{p(x_{i-1}, x_i)}{p(x_i, x_{i-1})} \qquad \text{where } x_0 = a \text{ and } x_n = x$$

whenever $\prod_{i=1}^{n} p(x_i, x_{i-1}) > 0$

$\mu(x)$ is well defined because of the loop condition.



$$\mu(x)\, p(x,y) = \prod_{i=1}^{n} \frac{p(x_{i-1}, x_i)}{p(x_i, x_{i-1})}\, p(x,y)\, \frac{p(y,x)}{p(y,x)}$$

$$= \mu(y)\, p(y,x)$$

Nice, huh?

Theorem (More intuitive way of defining the stationary measure)

Let x be recurrent, $T =$ 1st return time to x

$$\mu_x(y) = E_x[\# \text{ of visits to } y \text{ before 1st return to } x]$$

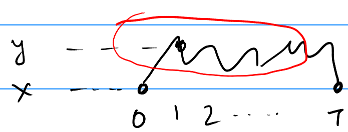
$$= E_x\left[\sum_{n=0}^{T-1} 1_{\{X_n = y\}}\right] = E_x\left[\sum_{n=0}^{\infty} 1_{\{X_n = y\}} 1_{\{n < T\}}\right]$$

$$= \sum_{k=0}^{\infty} P(X_k = y, T > k)$$

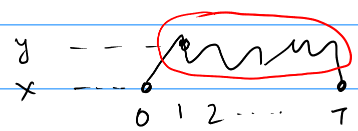
Set $\mu_x(x) = 1$

Called "cycle trick"

$\mu_x(y) =$ expected # of visits in $\{0, \dots, T-1\}$



$\equiv$



$= \sum \mu_x(z)\, p(z,y)$ = "expected # of visits to y in $\{1, \dots, T\}$"

(which I have never quite understood).

Pf If $y \neq x$ then drop $k=0$ term, and include $k=T$ term

key step

You can do this because $P_x(X_0=y, T>0)=0$ $\quad P_x(X_k=y, T=k)=0$

$$\mu_x(y) = \sum_{k=1}^{\infty} P_x(X_k=y, T \geq k) = \sum_{k=1}^{\infty} \sum_z P_x(X_{k-1}=z, X_k=y, T \geq k)$$

$$= \sum_{k=1}^{\infty} \sum_z E_x\left[1_{\{X_{k-1}=z\}} 1_{\{T \geq k\}} P_z(X_1=y) \right]$$

by Markov & using $X_{k-1}=z,\ T \geq k \in \mathcal{F}_{k-1}$.

$$= \sum_z \sum_{k=1}^{\infty} P(X_{k-1}=z, T \geq k)\, p(z,y) = \sum_z \sum_{k=0} P(X_k=z, T>k)\, p(z,y)$$

$$= \sum_z \mu_x(z)\, p(z,y)$$

If $y=x$ still need to verify

$$\mu_x(x) = 1 \quad = \sum_y \mu_x(y)\, p(y,x)$$

$$= \sum_y \sum_{k=0}^{\infty} P_x(X_k=y, T>k)\, p(y,x)$$

$$= \sum_{y \neq x} \sum_{k=0}^{\infty} P_x(X_k=y, T>k, X_{k+1}=x)$$

$$= \sum_{k=0}^{\infty} \sum_{y \neq x} P_x(X_k=y, T=k+1)$$

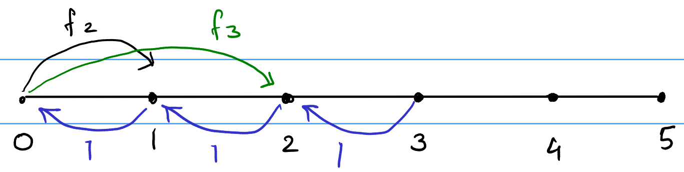
$$= \sum_{k=0}^{\infty} P_x(T=k+1) = 1 \quad \text{since } P_x(T>0)=1$$

So we have done two things:

1) Determine when reversible measures exist
2) Finding when stationary measures exist.

Ex: (Renewal Chain)

$$S = \{0, 1, 2, \ldots\}$$



$$p(0,j) = f_{j+1} \qquad \sum_{j=1}^{\infty} f_j = 1$$

$$p(j, j-1) = 1$$

There are arrivals of renewals T_i is the i^{th} arrivals time

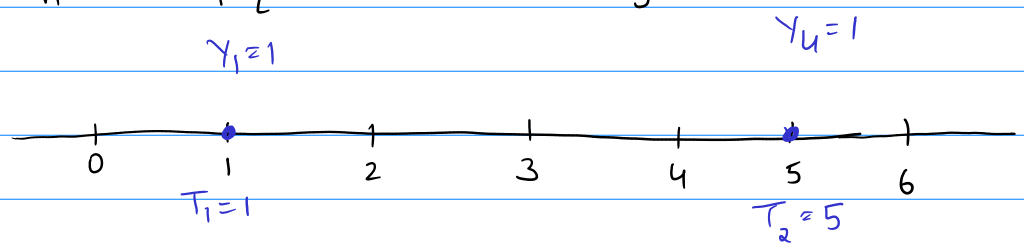
Interarrival times are iid.

$$T_0 = i_0 \qquad T_k = T_{k-1} + \xi_k$$

Let $Y_m \in \{0, 1, 2, \ldots\}$

$$Y_m = \begin{cases} 1 & \text{if } m \text{ is an arrival time.} \\ 0 & \text{otherwise} \end{cases}$$

$$X_n = \inf\{m - n : Y_m = 1\}$$

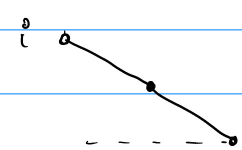


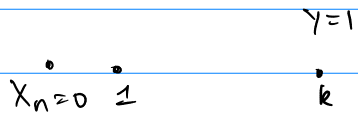
X_n is the time until the next arrival past n

$P(T_k < \infty) = 1$ So we will always recur to 0.

☆ Ex: Show that X_n is a Markov chain.

If $X_n = i > 0$ $\quad X_{n+1} = i - 1$





If $\xi = k$ then $X_{n+1} = k - 1$ and so on

☆ If $f_K > 0$ but $f_k = 0$ $\forall\, k > K$ then

$\{0, 1, \dots\}$ is not irreducible

$\{0, 1, \dots K-1\}$ is irreducible. (The biggest jump in X is $K-1$)

If you start at 0. You will hit a point j only

if the first jump is to a point $k \geq j$.

So $\mu_0(j) = E_0\left[\sum_{m=0}^{T-1} 1_{\{X_m = j\}}\right] = E\left[1_{\{\xi \geq j+1\}}\right]$

$= P(\xi > j)$

$\sum_{j=0}^{\infty} \mu_0(j) = \sum_{j=0}^{\infty} P(\xi > j) = E[\xi]$

So if $E[\xi] < \infty$ then $\pi(j) = \dfrac{P(\xi > j)}{E\xi}$ is a stationary distribution

Theorem: If p is irreducible and recurrent, then μ the stationary measure exists and is unique.

This is theorem 6.5.3 in the printed version of the book.

Pf: let ν, μ be stationary measures $a \in S$.

Recall that $\mu_a(z) = \mathbb{E}_a\left[\sum_{i=0}^{T-1} 1_{\{X_i = z\}}\right] = \sum_{k=0}^{\infty} P(X_k = z, T > k)$

$$\nu(z) = \sum_y \nu(y) p(y,z)$$ ← This computation is key

$$= \nu(a) p(a,z) + \sum_{y \neq a} \nu(y) p(y,z)$$

$$= \nu(a) P(X_1 = z, T > 1) \qquad + \sum_{y \neq a} \nu(y) p(y,z)$$ ↳ expand

separate out $y=a$ term

$$= \nu(a) p(a,z) + \sum_{y \neq a} \nu(a) p(a,y) p(y,z)$$

$$+ \sum_{y \neq a} \sum_{x \neq a} \nu(x) p(x,y) p(y,z)$$

$$= \nu(a) P(X_1 = z, T > 1) + \nu(a) P(X_2 = z, T > 2)$$

$$+ P_\nu(X_2 = z, T > 2)$$

$$= \nu(a) P_a(X_1 = z, T > 1) + \sum_{m=1}^{n} \nu(a) P(X_m = z, T > m)$$

$$+ P_\nu(X_m = z, T > m) \qquad (\star 2)$$

One wants to show that since the chain is recurrent, we have

$$\lim_{m\to\infty} P_a(T>m) = 0 \quad \text{and} \quad \text{thus}$$

$$\lim_{m\to\infty} P_\nu(X_m = z, T>m) = 0$$

Going back to ★2 and letting $n\to\infty$ and

using $P_\nu(X_m = z, T>m)$

$$\nu(z) \geq \nu(a)\sum_{n=1}^{\infty} P_a(X_n = z, T>n)$$

$$= \nu(a)\,\mu_a(z)$$

stationarity of ν

$$\nu(a) = \sum_x \nu(x)\,p^n(x,a) \geq \nu(a)\sum \mu_a(x)\,p^n(x,a)$$

$$= \nu(a)\,\mu_a(a) = \nu(a)$$

stationarity of μ

So all the inequalities are equalities and

$$\nu(x) = \nu(a)\mu_a(x) \quad \forall x \text{ st } p^n(x,a)>0 \quad (\exists\, n \text{ st})$$

Now using irreducibility and varying n, we get our result, which is that

$$\nu(x) \sim c\,\mu_a(x) \quad \forall x\in S.$$

This also shows that

$$\lim_{n\to\infty} P_\nu(X_n = z, T > m) = 0$$

Remark: It's hard to directly say this because one would need some sort of uniformity in a for $\nu(a) P_a(T>m) \xrightarrow[m\to\infty]{} 0$. This is because $\nu(a)$ could be some ∞ measure. If $\sum \nu(a) < \infty$ then we could show this.

Le Gall also uses a similar proof.

Stationary Distributions

π is called a stationary distribution if

$$\pi p = \pi$$

and $\quad \sum \pi = 1 \quad$ (π is a probability meas)

You can take and μ stationary and divide by $\mu(S)$ if $\mu(S) < \infty$ to get a $\pi = \dfrac{\mu}{\mu(S)}$

From previous if π exists, it is unique.

<u>6.5.4</u> Stationary distributions.

$\pi p = \pi$ and $\sum_y \pi(y) = 1$.

If π exists, then ~~if π~~ all

$\{y : \pi(y) > 0\}$ are recurrent.

$$\underbrace{\sum_x \pi(x) \sum_{n=1}^{\infty} p^n(x,y)}_{\text{Expected \# of visits to } y \text{ under the } P_\pi \text{ measure}} = \sum_{n=1}^{\infty} \underbrace{\pi(y)}_{>0} = +\infty$$

Expected # of visits to y.

under the P_π measure

Remains to show $E_y[N_y] = +\infty$

But $\displaystyle\sum_{n=1}^{\infty} p^n(x,y) = \frac{\rho_{xy}}{1-\rho_{yy}}$

$$\Rightarrow \infty = \frac{\sum \pi(x)\, \rho_{xy}}{1-\rho_{yy}} \leq \frac{1}{1-\rho_{yy}}.$$

$\Rightarrow$ $\rho_{yy} = 1$ and y is recurrent.

6.55 If p irred. & recurr. with π stationary

then

$$\pi(x) = \frac{1}{\underbrace{E_x T_x}_{\substack{\text{average return time} \\ \text{starting at } x.}}}$$

I think Durrett's and my version of the proof is a bit clearer.

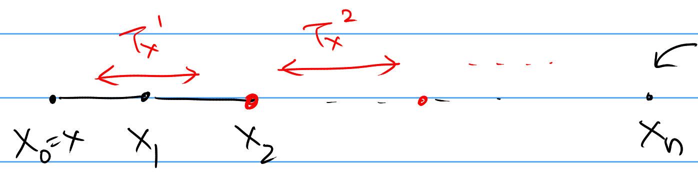
$$\sum_y \mu_x(y) = \sum_{k=0}^{\infty} P_x(X_k = y, T_x > k)$$

$$= \sum_{k=0}^{\infty} P_x(T_x > k) = E_x[T_x]$$

Then, $\pi(y) = c\,\mu_x(y) \Rightarrow \sum_y \pi(y) = 1 = c\,E_x[T_x]$

In particular $\pi(x) = c\,\mu_x(x) = c \cdot 1 = \dfrac{1}{E_x[T_x]}$

Another way to view will be through the ergodic thm.



$$\frac{\#\text{ of times } X_k = x}{n} \approx \pi(x)$$

$$T_x^1 + (T_x^2 - T_x^1) + \cdots + (T_x^{N_x} - T_x^{N_x - 1}) = n$$

$\Rightarrow \quad E_x[N_x]\, E_x[T_x] = n \qquad$ But $\dfrac{E_x[N_x]}{n} = \pi(x)$

which gives the result.

Positive recurrence $\iff E_x[T_x] < \infty$

Null and positive recurrence

(i) Some x is +ve recurrent

(ii) $\exists$ a stationary distribution

(iii) All states are positive recurrent.

If even 1 +ve recurrent state, then all states are +ve recurrent.

(i) $\Rightarrow$ (ii) x is +ve recurrent

$$\pi(y) = \frac{\sum_{n=0}^{\infty} \mathbb{P}_x(X_n = y, T_x > n)}{E_x T_x} = \frac{\mu_x(y)}{E_x T_x}$$

is a stationary (from previous). distribution

(ii) $\Rightarrow$ (iii) Clear: $\pi(y) > 0 \ \forall y$

$\Rightarrow \pi(y) = \dfrac{1}{E_y T_y}$. done.

(iii) $\Rightarrow$ (i) obvious.

So positive recurrence is a <u>class property</u>.

OK. Recall RW.

$$\mu(x) = 1 \quad \text{is a stationary meas.}$$

(why $p(x,y) = f(y-x)$).

If p irred & has st. ν with $\sum_x \nu(x) = \infty$ then p is <u>not</u> positive recurrent.

↳ Suppose $\exists\, y \quad E_y T_y < \infty$

$$\Rightarrow \quad \mu_y(z) = \frac{\sum_{n=0}^{\infty} \mathbb{P}_y(X_n = z, T > n)}{E_y T_y} = \pi(z)$$

is a stationary distribution!

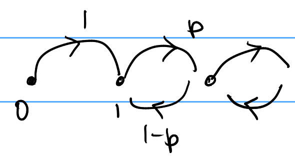
Then 6.5.3 $\Rightarrow$

$$\nu(x) = c\,\pi(x)$$

uniqueness up to constants

$$\Rightarrow \quad \sum \nu(x) = c < \infty \qquad \blacksquare$$

Example (from Le Gall):



$$\mu(k) = \left(\frac{p}{1-p}\right)^{k-1} \quad k \geq 1$$

$$\mu(0) = 1-p$$ is a stationary measure.

$\mu(\mathbb{Z}^+) < \infty$ if $p < 1/2$ (+ve recurrent)

$\mu(\mathbb{Z}^+) = +\infty$ if $p = 1/2$ (Null recurrent 1D RW)

$\mu(\mathbb{Z}^+) = +\infty$ if $p > 1/2$ (Transient, SLLN)

So RW not <u>positive</u> recurrent.

★ Ex: What about RW on a finite graph?

Recall our definition.

$$\mu(i) = \sum_j a_{ij} = \text{\# of out edges of vertex } i.$$

Showed that μ is reversible wrt to

$$p(i,j) = \frac{a_{ij}}{\mu(i)}$$

Using $a_{ij} = a_{ji}$!

If G is connected, can you get a LB for $\mu(i)$?

$\mu(i) \geq 1$ (at least one out edge)

$\Rightarrow$ G is +ve recurrent iff

________________ ?

Birth and death chain

Ex: (Satisfies cycle condition)

$$\mu(x) = \prod_{k=1}^{x} \frac{p_{k-1}}{q_k}$$

This was reversible. So this has positive recurrence iff $\sum_x \mu(x) < \infty$.

But recall

$$\sum_{x=0}^{\infty} \prod_{k=1}^{x} \frac{q_k}{p_k} = \infty$$

$\Leftrightarrow$ recurrence.

So when is null recurrence?

~~So~~, when

$$\sum_{x} \prod_{k=0}^{x} \frac{p_{k-1}}{q_k} = \infty$$

and

$$\sum_{x} \prod_{k=0}^{x} \frac{q_k}{p_k} = \infty$$

Ex: Can you give me a good example where this is true?

Thm: If p is irreducible and has a stationary distribution π then any other stationary meas. is a multiple of π.

Pf: Follows from previous, but Durrett gives an entropy proof thats not very illuminating

Exercises:

5.5.1 Find stationary distrib for the Bernoulli-Laplace model of diffusion.

5.5.2 $w_{xy} = P_x(T_y < T_x)$ $\mu_x(y) = \frac{w_{xy}}{w_{yx}}$

I have proved this $\rightarrow$ 5.5.5 If p is irred and has a μ st $\sum \mu = \infty$ then p is not +ve recurrent.

5.5.6 Show that for an SRW, the expected # of visits to k between successive visits to 0 is 1 $\forall k$.

5.5.8 $M/M/\infty$ queue. Find stationary distribution.

(6.5.7) If p is irred. and has stationary distribution π then any other stationary meas is a multiple of π.

↳ Essentially follows from

6.5.4 : If π is a stationary distribution, then all states with $\pi(y) > 0$ are recurrent.

But p is irred, so $\pi(y) > 0 \;\forall y \Rightarrow$ p is recurrent.

6.5.3 If p is irred and recurrent Then stationary measures are unique up to constants!